Sampling Archimedean copulas

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1 Nonnested Archimedean copulas

- **Kimberling:** $\varphi^{-1}$ c.m. (i.e. $(-1)^k \frac{d^k}{dt^k} \varphi^{-1}(t) \geq 0$)

  \[\Leftrightarrow \varphi^{-1}[\varphi(u_1) + \cdots + \varphi(u_d)] \text{ a copula.}\]

- **Sampling** $\varphi^{-1}[\varphi(u_1) + \cdots + \varphi(u_d)]:$
  - Via **Conditional distribution method** (density known).
  - Via **Laplace Stieltjes transforms**.

- **Bernstein:** $\varphi^{-1}$ c.m. and $\varphi^{-1}(0) = 1$

  \[\Leftrightarrow \varphi^{-1} = \mathcal{LS}(F_V(x)), \quad \text{supp}(F_V) \subset [0, \infty).\]
1.1 Marshall and Olkin’s algorithm

Algorithm (Marshall, Olkin)

(1) Sample \(V \sim F_V\).

(2) Sample i.i.d. realizations \(X_i \sim U[0, 1], i \in \{1, \ldots, d\}\).

(3) Return \((U_1, \ldots, U_d)\), where \(U_i = \varphi^{-1}(-\log(X_i)/V)\).

\(\Rightarrow\) Fast and easy (if \(F_V\) is easy to sample).

Examples for \(F_V\):

- AMH: \(\sum_{k=1}^{n} (1 - \vartheta)\vartheta^{k-1}, n \in \mathbb{N}, \vartheta \in [0, 1) \Rightarrow \text{Geometric}\).

- Frank: \(\sum_{k=1}^{n} \frac{(1-e^{-\vartheta})^k}{k^\vartheta}, n \in \mathbb{N}, \vartheta \in (0, \infty) \Rightarrow \text{Logarithmic}\).

- Joe: \(\sum_{k=1}^{n} (-1)^{k+1} \left(\frac{1}{k}\right)^{\vartheta}, n \in \mathbb{N}, \vartheta \in [1, \infty)\).
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- Clayton: $\Gamma(1/\vartheta, 1), \vartheta \in (0, \infty) \Rightarrow \text{Gamma}.$

- Gumbel: $S(1/\vartheta, 1, (\cos(\frac{\pi}{2\vartheta}))^{\vartheta}, 0; 1), \vartheta \in [1, \infty) \Rightarrow \text{Stable}.$

**Theorem**
Let $\varphi^{-1} = \mathcal{L}S(F_V)$ and $F(x) = \sum_{k=0}^{\infty} y_k \mathbb{1}_{[x_k, \infty)}(x)$ with $0 < x_0 < x_1 < \ldots$ and $y_k \geq 0, k \in \mathbb{N}_0$, with $\sum_{k=0}^{\infty} y_k = 1$. Then

$$F_V \equiv F \iff \varphi^{-1}(t) = \sum_{k=0}^{\infty} y_k e^{-x_k t}. $$
2 Nested Archimedean copulas

• Now: \( C'(u) = \varphi^{-1}[\varphi(u_1) + \varphi(\varphi_1^{-1}[\varphi_1(u_2) + \varphi_1(u_3)])] \)
  - Fully vs. partially nested Archimedean copulas.
  - Nonexchangeable (max. \( d - 1 \) different pairwise dependencies).
  - Allows for modeling different industry sectors, regions, etc.

• McNeil: Nesting condition \((\varphi \circ \varphi_1^{-1})' \) c.m. \( \Rightarrow C'(u) \) is a copula.

• \((\varphi \circ \varphi_1^{-1})' \) c.m. \( \Rightarrow \varphi_0^{-1}(t; v) = e^{-v\varphi \circ \varphi_1^{-1}(t)} \) is a generator inverse for every \( v \in (0, \infty) \) with corresponding inner \( F_V = \mathcal{L}S^{-1}(\varphi_0^{-1}) \).
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2.1 McNeil’s algorithm

Algorithm (McNeil)
(1) Sample $X_1 \sim U[0, 1]$.

(2) Sample $V \sim F_V = \mathcal{L}S^{-1}(\varphi^{-1})$, $(X_2, X_3) \sim C(u_2, u_3; \varphi_{0,1}^{-1}(\cdot; V))$.

(3) Return $(U_1, U_2, U_3)$, where $U_i = \varphi^{-1}(-\log(X_i)/V)$.

Proof
Solving w.r.t. the $X$’s and conditioning under $V$ leads to

$$
\mathbb{P}(U \leq u) = \int_0^\infty e^{-v \varphi(u_1)} \varphi_{0,1}^{-1}[\varphi_{0,1}(e^{-v \varphi(u_2)}) + \varphi_{0,1}(e^{-v \varphi(u_3)})] \, dF_V(v)
$$

$$
= \int_0^\infty e^{-v \varphi(u_1)} e^{-v \varphi(\varphi_1^{-1}[\varphi_1(u_2) + \varphi_1(u_3)])} \, dF_V(v)
$$

$$
= \varphi^{-1}[\varphi(u_1) + \varphi(\varphi_1^{-1}[\varphi_1(u_2) + \varphi_1(u_3)])].
$$
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Examples:

- Nested **Gumbel** copula:
  
  \[ \varphi_{0,1}^{-1}(t; v) = e^{-vt^{\vartheta/\vartheta_1}} \Rightarrow \varphi_{0,1}^{-1}(t/c; v) \text{ with } c = v^{\vartheta_1/\vartheta} \text{ belongs to the Gumbel family again.} \]

  \[ \text{Sample } C(u_2, u_3; \varphi_{0,1}^{-1}(\cdot/c; V)) \text{ (involves Stable distribution).} \]

- Nested **Clayton** copula:
  
  \[ \varphi_{0,1}^{-1}(t; v) = e^{-v((1+t)^{\vartheta/\vartheta_1}-1)} \Rightarrow \text{Rejection algorithm (only for large enough } \vartheta). \]

Questions:

- Which **(classes of) generators** can be used to build a nested structure?
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• Corresponding sampling strategies?

• Fast in large dimensions?

2.2 A general nesting result

• Power families \((d = 2)\):
  
  – **Inner power families**: \(\tilde{\varphi}(t) = \varphi(t^\alpha)\) for any \(\alpha \in (0, 1]\).
  
  – **Outer power families**: \(\tilde{\varphi}(t) = \varphi(t)^\beta\) for any \(\beta \in [1, \infty)\).

\(\Rightarrow\) Can they be nested and easily sampled?

• For inner power families, the sufficient nesting condition is usually not fulfilled (e.g. take AMH based on \(\vartheta = 1/2\)).
Theorem

(a) For an Archimedean generator \( \varphi \) with c.m. inverse,
\[
\tilde{\varphi}(t) = (c + \varphi(t))^{\vartheta} - c^{\vartheta}
\]
has a c.m. inverse for any \( \vartheta \in [1, \infty) \) and \( c \in [0, \infty) \).

(b) The sufficient nesting condition holds for any \( \vartheta \leq \vartheta_1 \) and the inner generator inverse is given by
\[
e^{-v\tilde{\varphi}\circ\tilde{\varphi}_1^{-1}(t)} = e^{-v((c^{\vartheta_1} + t)^{\vartheta_1}/\vartheta_1 - c^{\vartheta})}.
\]

- Families \# 1, 4, 12, 13 and 19 of Nelson (1998) fall under this setup.

- The inner \( F_V \) is an exponentially tilted Stable distribution \( (c = 1 \Rightarrow \text{Clayton}) \).

- For outer power families \( (c = 0) \) we obtain:
  They extend to \( d > 2 \) and they can be nested.
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- The inner $F_V$ is a Stable distribution, independent of the family on which the outer power family is built!

- For the outer $F_V$, the following algorithm generates a
  \[ \tilde{V} \sim F_{\tilde{V}} = \mathcal{L}S^{-1}(\hat{\varphi}^{-1}), \]  where \( \hat{\varphi}(t) = \varphi(t)^{\vartheta}, \vartheta \in [1, \infty). \)

Algorithm

1. Sample \( V \sim F_V = \mathcal{L}S^{-1}(\varphi^{-1}). \)
2. Sample \( S \sim S(1/\vartheta, 1, (\cos(\frac{\pi}{2\vartheta}))^{\vartheta}, 0; 1). \)
3. Return \( SV^{\vartheta}. \)

Proof

\[
(\mathcal{L}S(F_{SV^{\vartheta}}))(t) = \int_0^\infty \int_0^\infty e^{-tv^{\vartheta}s} f_S(s) \, ds \, dF_V(v) \\
= \int_0^\infty e^{-t(v^{\vartheta})^{1/\vartheta}} dF_V(v) = (\mathcal{L}S(F_V))(t^{1/\vartheta}) = \hat{\varphi}^{-1}(t). \]

\[ \square \]
Knowing $F_V = \mathcal{L}S^{-1}(\varphi^{-1})$, we can build and easily sample an outer power nested Archimedean copula based on $\varphi$.

**Theorem**

Kendall’s tau $\tau_{\tilde{\varphi}}$, belonging to the copula $\tilde{C}$ generated by $\tilde{\varphi}(t) = \varphi(t)^{\vartheta}$, $\vartheta \in [1, \infty)$, is given by

$$
\tau_{\tilde{\varphi}} = 1 - \frac{1}{\vartheta}(1 - \tau_{\varphi}).
$$

**Example: A nested outer power Clayton copula**

- Outer power family build on Clayton’s generator $\varphi(t) = t^{-\vartheta_c} - 1$.

  $$
  \tau_{\tilde{\varphi}} = 1 - \frac{1}{\vartheta} \frac{2}{\vartheta_c + 2}, \quad \lambda_{L,\tilde{\varphi}} = 2^{-1/(\vartheta_c \vartheta)} \text{ and } \lambda_{U,\tilde{\varphi}} = 2 - 2^{1/\vartheta}
  $$
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- Scatterplot matrix for a **fully nested outer power Clayton copula**
  \((\vartheta_c = 1, \vartheta = 1.1, \vartheta_1 = 1.5)\):
2.3 Special nesting results

Discrete distributions

- For AMH, Frank and Joe: outer $F_V$ discrete.
  $\Rightarrow$ What about the inner $F_V$’s?

**Theorem**

If $\varphi$ and $\varphi_1$ denote Joe’s generators with $\vartheta \leq \vartheta_1$, then $\varphi_{0,1}^{-1}(t; v)$, $v \in \mathbb{N}$, has Laplace Stieltjes inverse $F_V(x) = \sum_{k=1}^{\infty} y_k \mathbb{1}_{[x_k, \infty)}(x)$ with

$$x_k = k \text{ and } y_k = \sum_{j=0}^{v} (-1)^{j+k} \binom{v}{j} \binom{j \vartheta / \vartheta_1}{k}, \quad k \in \mathbb{N}.$$ 

- Precalculation **numerically complicated**: Runtime (dependence on $v$ sample) and errors ($F_V(k) = 0$, $k < v$ and $F_V(v) = (\vartheta / \vartheta_1)^v$).
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⇒ Not adequate for sampling purposes!

Idea

Use the generating function for the inner $F_V$:

$$g(x) = (1 - (1 - x)^{\nu/\vartheta_1})^\vartheta.$$ 

Lemma (Devroye)

Let $g, g_1, g_2$ be g.f.’s such that $g(t) = g_1(g_2(t))$ and let $N$ and $X$ have g.f. $g_1$ and $g_2$, respectively, then

$$Y = \sum_{i=1}^{N} X_i \text{ has generating function } g,$$

where the $X_i$’s are i.i.d. copies of $X$. 
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Algorithm

(1) Precalculate the outer $F_V$ once $(y_k = (1/\vartheta)(-1)^{k-1}, k \in \mathbb{N})$.

(2) Precalculate $F$ corresponding to the g.f. $1 - (1 - x)^{\vartheta/\vartheta_1}$ once
\[ (y_k = (\vartheta/\vartheta_1)(-1)^{k-1}, k \in \mathbb{N}) . \]

(3) Sample $V \sim F_V$.

(4) Sample i.i.d. $X_i \sim F$, $i \in \{1, \ldots, V\}$ and build $\sum_{i=1}^{V} X_i$ (a sample of the inner $F_V$).

(5) Proceed as before.

- For Frank’s nested family: Similar.
- For AMH’s nested family: Inner $F_V$ also a geometric distribution (no precalculation).
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- Scatterplot matrix for a fully nested Joe copula ($\vartheta = 1.2, \vartheta_1 = 1.6$):
Mixed families

- Is it possible to mix different families?
- Examples from Nelson (1998): The sufficient nesting condition holds for 7 family combinations including \( (\text{AMH,Clayton}) \) for \( \vartheta \in [0, 1), \vartheta_1 \in [1, \infty) \).

A nested \( (\text{AMH,Clayton}) \) copula

- Parameters \( \vartheta = 0.8 \) and \( \vartheta_1 = 2 \).
- Outer \( F_V \): Known geometric distribution.
- Inner \( F_V \): \( \varphi_{0,1}^{-1}(t; v) = ((1 - \vartheta)(1 + t)^{1/\vartheta_1 + \vartheta})^{-v} \Rightarrow \text{Inner } F_V \text{ not explicitly known.} \)
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Idea

Apply numerical inversion of Laplace transforms and numerical rootfinding to obtain a realization from

\[ F_V(x) = (\mathcal{L}^{-1}(\varphi^{-1}(t)/t))(x), \; x \in [0, \infty). \]

- Methods:
  - Fixed Talbot algorithm (contour deformation in the Bromwich integral).
  - Gaver Stehfest and Gaver Wynn rho algorithm (discrete analog of the Post-Widder formula).
  - Laguerre series algorithm (approximation via Laguerres series).

- Comparison in the known nonnested Clayton case ($\vartheta = 0.8$):
Parameter calibration according to the precision requirement

\[
\max_{x \in P} \left| \frac{F_V(x) - \tilde{F}_V(x)}{F_V(x)} \right| < 0.0001 \text{ for } P = \{0.001, 0.002, \ldots, 10\}.
\]

for an approximation \( \tilde{F}_V \) to \( F_V \).

2.4 Simulation results

- \( \chi^2 \)-test based on hypercubes of a partition into 5 parts for each dimension.
- Maximal (\textsc{MXDEV}) and mean (\textsc{MDEV}) deviations of the probabilities of falling in the hypercubes.
- Matrix of pairwise sample versions of Kendall’s tau.
- Visual check of plots of outer and inner \( F_V \)’s.
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For the nested (AMH, Clayton) copula

Based on 500,000 observations, \( \vartheta = 0.8 \) and \( \vartheta_1 = 2 \), using the Fixed Talbot algorithm:

- \( p \)-value = 0.000000000000 (\( \chi^2 \)-test).
- \( \text{MXDEV} = 0.000555 \) and \( \text{MDEV} = 0.000084 \).
- \( \hat{\tau}_\vartheta \in \{0.2346, 0.2343\} \) (\( \tau_\vartheta = 0.2337 \)) and \( \hat{\tau}_{\vartheta_1} = 0.5009 \) (\( \tau_{\vartheta_1} = 0.5 \)).
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- Plots for the outer (left) and inner (right) distribution functions $F_V$:

![Diagrams showing plots for outer and inner distribution functions with different parameter values.](image-url)
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- Scatterplot matrix for this fully nested (AMH, Clayton) copula ($\vartheta = 0.8$, $\vartheta_1 = 2$):

![Scatterplot matrix for fully nested (AMH, Clayton) copula](image)
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Runtimes

For 500,000 observations of \( \varphi^{-1}[\varphi(u_1) + \varphi(\varphi_1^{-1}[\varphi_1(u_2) + \varphi_1(u_3)])] \), we obtain:

- (AMH, Clayton): 37.67s.
- Gumbel: 2.22s.
  - Outer power Clayton: 2.71s.
- Joe: 1.50s.
  - Frank: 1.60s.
  - AMH: 0.59s.