Modelling Dependence of Time Series with Copulae

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Log Returns of BMW and Bayer

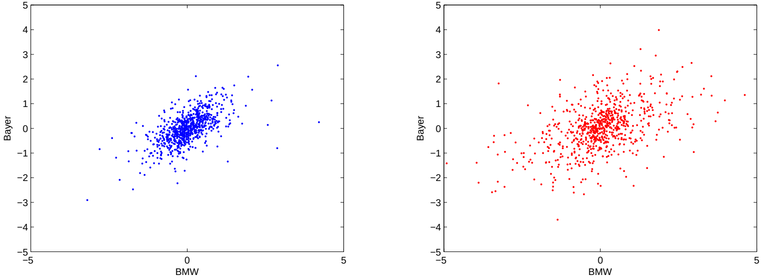

Figure 1: Standardized log returns, BMW and Bayer, 19920101-19950102 (left) and 19960101-19990101 (right).

Dimensionality

In d-dimension

1. Elliptical Copulae: correlation matrix with $\frac{d(d-1)}{2}$ parameters
2. Archimedean Copulae: 1 parameter

Applied Copulae

1. Elliptical Copulae
   - Gaussian
   - *t*-Student
2. Archimedean Copulae
   - Gumbel
   - Cooka-Johnson
   - Frank
3. Mixture Copulae
4. Hierarchical Archimedean Copulae

Outline

1. Motivation ✓
2. Mixture copulae
3. Hierarchical Archimedean copulae
4. Value-at-Risk and Expected Shortfall with copulae

Mixture Copula

Let C_1, ..., C_K be d-dimensional copulae and w_1, ..., w_K positive weights so that $w_1 + \ldots + w_K = 1$.
Mixture copula:

$$C_{mix}(u_1, \ldots, u_d) = \sum_{i=1}^{K} w_i C_i(u_1, \ldots, u_d).$$

Mixture Copula

Joint density of $(X_1, \ldots, X_d)^\top$

$$f(x_1, \ldots, x_d; \psi) = \sum_{i=1}^{K} w_i f_i(x_1, \ldots, x_d; \theta_i),$$

where $\psi = (w_i, \theta_i)_{i=1}^{K}$ and

$$f_i(x_1, \ldots, x_d; \theta_i) = c_i\{F_1(x_1), \ldots, F_d(x_d); \theta_i\} \prod_{j=1}^{d} g_j(x_j),$$

where g_j, $j = 1, \ldots, d$, are densities of marginal distributions.

Description of the Data Set

1. Warsaw Stock Exchange, http://www.gpw.pl/
2. Polish stocks: KGHM Polska Miedź SA (metallurgical industry), PKN Orlen SA (fuel), Telekomunikacja Polska SA (telecommunications), ComputerLand SA (computer science), Orbis SA (tourism), Bank Pekao SA (bank)
3. 1001 closing prices (1000 log-returns)
4. time period: 02.01.2003 – 18.12.2006

Example

$$C_{mix}(u_1,\dots,u_6) = wC_{Frank}(u_1,\dots,u_6) + (1-w)C_{Cook}(u_1,\dots,u_6),$$

where $\hat{\theta}_{Frank} = 2.5594$, $\hat{\theta}_{Cook} = 1.3063$, $\hat{w} = 0.8156$.

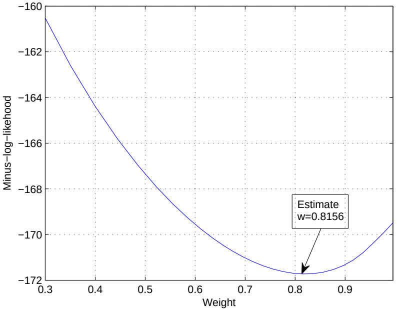

Figure 2: Parameters estimation of mixture copula with Expectation-Maximization method.

General Multivariate Archimedean Copula

Let $\varphi(t) : [0,1]^d \mapsto [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(0) = \infty$ and $\varphi(1) = 0$. Let φ^{-1} denote the inverse of φ. For all $n \geq 2$ the function $C : [0,1]^d \mapsto [0,1]$ given by

$$C(u_1, u_2, \ldots, u_d) = \varphi^{-1}\{\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_d)\}$$

is d-copula if and only if φ^{-1} is completely monotone on $[0,\infty)$.

Hierarchical Archimedean Copulae

A hierarchical Archimedean copula joins two or more ordinary bivariate or higher-dimensional Archimedean copulae by another Archimedean copula.

The fully nested HAC

$$C(u_1, u_2, u_3, u_4) = C_{31}[u_4, C_{21}\{u_3, C_{11}(u_1, u_2)\}]$$
$$= \varphi_{31}^{-1}[\varphi_{31}\{u_4\} + \varphi_{31}\{\varphi_{21}^{-1}(\varphi_{21}[u_3] + \varphi_{21}[\varphi_{11}^{-1}\{\varphi_{11}(u_1) + \varphi_{11}(u_2)\}])\}]$$

The partially nested HAC

$$C(u_1, u_2, u_3, u_4) = C_{21}\{C_{11}(u_1, u_2), C_{12}(u_3, u_4)\}$$
$$= \varphi_{21}^{-1}(\varphi_{21}[\varphi_{11}^{-1}\{\varphi_{11}(u_1) + \varphi_{11}(u_2)\}] + \varphi_{21}[\varphi_{12}^{-1}\{\varphi_{12}(u_3) + \varphi_{12}(u_4)\}])$$

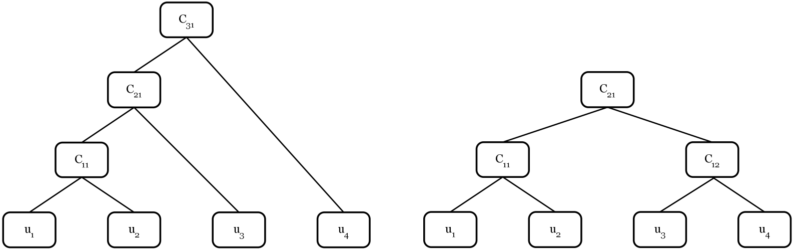

Figure 3: Fully (left) and partially nested (right) HAC.

Determining the Structure for $d = 3$

Consider $\{X_1, X_2, X_3\}$ and an archimedean copula C.
Grouping based on θ's:

1. estimate the parameters $\theta_{(1,2,3)}$, $\theta_{(1,2)}$, $\theta_{(1,3)}$, $\theta_{(2,3)}$
2. determine the best copula fit
   $\Delta = \min\{(\theta_{(1,2,3)} - \theta_{(1,2)}), (\theta_{(1,2,3)} - \theta_{(1,3)}), (\theta_{(1,2,3)} - \theta_{(2,3)})\}$
3. introduce a new variable Z by joining two selected variables with copula C
4. estimate the parameter of bivariate copula that joints variable Z and variable that left from the set $\{X_1, X_2, X_3\}$

Risk Measures

1. Value-at-Risk (negative)

$$VaR^X_{1-\alpha} = Q^X_\alpha = -q^{-X}_{1-\alpha},$$

   - $Q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) > \alpha\},$
   - $q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$

2. Expected Shortfall

$$ES^X_{1-\alpha} = E(X|X < VaR^X_{1-\alpha}).$$

Margins

The process $\{X_t\}_{t=1}^{T}$ of log-returns can be modeled with $GARCH(1,1)$

$$
\begin{aligned}
Y_t &= \mu + \sigma_t \varepsilon_t, \\
\sigma_t^2 &= c_0 + c_1 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + b_1 \sigma_{t-1}^2.
\end{aligned}
$$

The standardized innovations

$$\{\varepsilon_1, \ldots, \varepsilon_d\}$$

are independent with distribution function

$$F(\varepsilon_1, \ldots, \varepsilon_d) = C\{F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d)\},$$

where

- C is copula with parameter θ,
- ε_j have continuous marginal distributions F_j, $j = 1, \ldots, d$.

Risk Measures with Copulae

For a sample of log-returns $\{X_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$

1. specification of innovations $\hat{\varepsilon}_{j,t}$ by fitting $GARCH(1,1)$
2. specification of marginal distributions $F_{X_j}(x_j; \delta_j)$
3. specification of copula $C(u_1, \ldots, u_d; \theta)$
4. fit of the copula C
5. generation of n Monte Carlo data $X_{T+1} \sim C\{F_1(x_1), \ldots, F_d(x_d); \hat{\theta}\}$
6. generation of a sample of portfolio profits $L_{T+1}(X_{T+1})$
7. estimation of $\widehat{VaR}_{1-\alpha}$ and $\widehat{ES}_{1-\alpha}$.

Estimation of VaR and ES

$$\widehat{VaR}^{L}_{1-\alpha} = L_{(\lfloor \alpha n \rfloor + 1):n},$$
$$\widehat{ES}^{L}_{1-\alpha} = \frac{1}{\lfloor \alpha n \rfloor} \sum_{i=1}^{\lfloor \alpha n \rfloor} L_{i:n},$$

where L is Profit and Loss function

$$L_{t+1} = \sum_{i=1}^{d} S_{i,t}(\exp(X_{i,t+1}) - 1) \qquad (1)$$

and $X_{t+1} = \log S_{t+1} - \log S_t$.

Moving Window

- Specify the subsets of size $h = 250$: $\{u_{j,t}\}_{t=s-h+1}^{s}$ for $s = h, \ldots, T$.
- Obtain the sequence $\{\widehat{VaR}_{1-\alpha}^{j}\}_{j=1}^{T-h}$, $\{\widehat{ES}_{1-\alpha}^{j}\}_{j=1}^{T-h}$ and $\{\theta_j\}_{j=1}^{T-h}$.

Backtesting

The estimated VaR values are compared with true realizations $\{L_t\}$ of the Profit and Loss function. An *exceedance* occurs when L_t is smaller than $\widehat{VaR}^t_{1-\alpha}$.
The ratio of the number of exceedances to the number of observations gives the *exceedances ratio*:

$$\hat{p} = \frac{1}{T-h} \sum_{t=h+1}^{T} I_{\{L_t < \widehat{VaR}^t_{1-\alpha}\}}.$$

For Expected Shortfall we define

$$\hat{q} = \frac{1}{T-h} \sum_{t=h+1}^{T} I_{\{L_t < \widehat{ES}^t_{1-\alpha}\}}.$$

Kupiec test

H_0 : the exceedances ratio for Value-at-Risk is equal to α.
Likelihood ratio statistic:

$$LR = 2\log\{(1-\hat{p})^{T-h-N}\hat{p}^{N}\} - 2\log\{(1-\alpha)^{T-h-N}\alpha^{N}\} \stackrel{\mathcal{L}}{\rightarrow} \chi^2_1,$$

where N – number of exceedances, $\hat{p}$ – exceedances ratio.
We use 0.05 significance level for the Kupiec LR statistic.

Embrechts, Kaufmann and Patie Measure

$$V^{ES} = \frac{|V_1^{ES}| + |V_2^{ES}|}{2},$$

$$V_1^{ES} = \frac{\sum_{t=h+1}^{T} (L_t - \widehat{ES}_{1-\alpha}^{t}) I_{\{L_t < \widehat{VaR}_{1-\alpha}^{t}\}}}{\sum_{t=h+1}^{T} I_{\{L_t < \widehat{VaR}_{1-\alpha}^{t}\}}}$$

$D_t = L_t - \widehat{ES}_{1-\alpha}^{t}$, D^{α} – empirical α-quantile $\{D_t\}_{t=h+1}^{T}$.

$$V_2^{ES} = \frac{\sum_{t=h+1}^{T} D_t I_{\{D_t < D^{\alpha}\}}}{\sum_{t=h+1}^{T} I_{\{D_t < D^{\alpha}\}}}$$

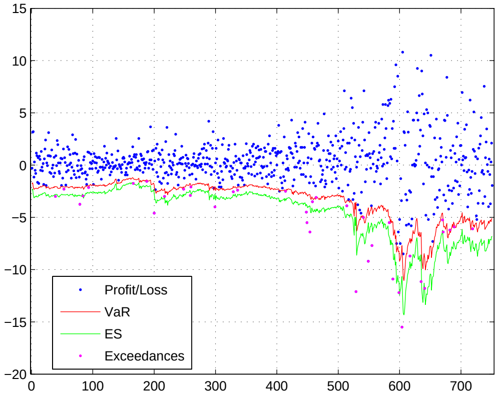

Figure 4: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with Cook-Johnson 3-copula and t-Student residuals.

Backtesting for 3-Copulae Archimedean and Elliptical

3-copula	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
Gaussian Residuals						
Gumbel	51	0.0679	4.5827	26	0.0346	0.9551
Cook-Johnson	38	0.0506	**0.0057**	18	0.0240	0.6727
Frank	42	0.0559	0.5355	20	0.0266	1.0035
Gauss	37	0.0493	0.0085	19	0.0253	0.8377
t-Student	37	0.0493	0.0085	19	0.0253	0.8148
t-Student Residuals						
Gumbel	53	0.0706	5.9666	21	0.0280	0.9022
Cook-Johnson	38	0.0506	**0.0057**	16	0.0213	**0.5331**
Frank	45	0.0599	1.4671	20	0.0266	0.9763
Gauss	40	0.0533	0.1649	19	0.0253	0.7510
t-Student	40	0.0533	0.1649	20	0.0266	0.7172

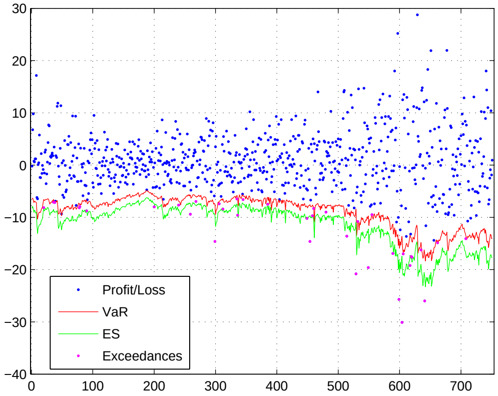

Figure 5: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with *t*-Student 6-copula and Gaussian residuals.

Backtesting for 6-Copulae Archimedean and Elliptical

6-copula	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
Gaussian Residuals						
Gumbel	48	0.0639	2.8245	28	0.0373	2.1217
Cook-Johnson	39	0.0519	0.0582	17	0.0226	1.0134
Frank	46	0.0613	1.8735	24	0.0320	2.0761
Gauss	38	0.0506	**0.0057**	19	0.0253	1.1093
t-Student	38	0.0506	**0.0057**	17	0.0226	0.9272
t-Student Residuals						
Gumbel	47	0.0626	2.3263	26	0.0346	2.0857
Cook-Johnson	36	0.0479	0.0682	17	0.0226	**0.5207**
Frank	43	0.0573	0.7970	26	0.0346	2.1775
Gauss	37	0.0493	0.0085	17	0.0226	0.9292
t-Student	37	0.0493	0.0085	15	0.0200	0.8213

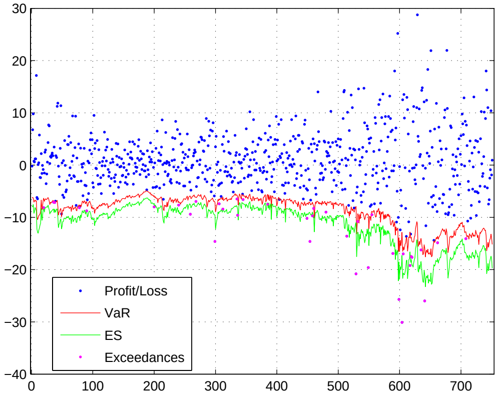

Figure 6: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with mixture copula of Frank 6-copula and Cook-Johnson 6-copula and t-Student residuals.

Backtesting for Mixture Copulae

Mixture	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
3-copula						
Gaussian Residuals						
Frank-CookaJohn.	35	0.0466	0.1863	18	0.0240	0.8709
Gumbel-CookaJohn.	40	0.0533	0.1649	19	0.0253	0.7083
Frank-Gumbel	45	0.0599	1.4671	21	0.0280	0.9460
t-Student Residuals						
Frank-CookaJohn.	37	0.0493	**0.0085**	18	0.0240	0.7194
Gumbel-CookaJohn.	40	0.0533	0.1649	18	0.0240	**0.6122**
Frank-Gumbel	47	0.0626	2.3263	20	0.0266	0.9198
6-copula. Gaussian Residuals						
Frank-CookaJohn.	39	0.0519	0.0582	19	0.0253	1.3383
6-copula. t-Student Residuals						
Frank-CookaJohn.	37	0.0493	**0.0085**	17	0.0226	1.0659

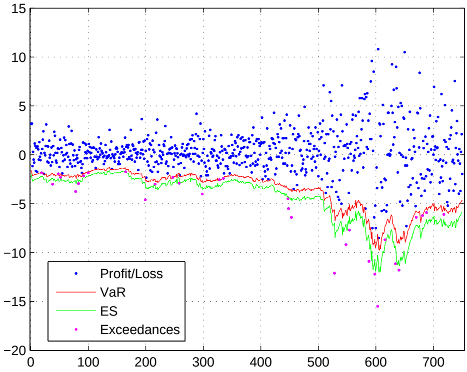

Figure 7: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with hierarchical archimedean copula with Gumbel generator and Gaussian residuals.

Backtesting for Hierarchical Archimedean Copulae

3-copula	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
Archimedean						
Gumbel	20	0.0266	10.282	14	0.0186	**0.3411**
Cook-Johnson	51	0.0680	4.6208	21	0.0280	0.6661
Frank	62	0.0826	14.198	33	0.0440	1.0851
Hierarchical Archimedean Copulae						
Gumbel	32	0.0426	**0.8916**	17	0.0226	0.5926
Cook-Johnson	30	0.0400	1.6900	16	0.0213	0.3885
Frank	58	0.0773	10.1834	32	0.0426	1.0459

Conclusion

The flexible and complex copula's structure can help in constructing more accurate multivariate models but makes the models more computational difficult and extends the time of obtaining results.

References

- X. Chen, Y. Fan, A. Patton
  Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates
- P. Embrechts, F. Lindskog, A. McNeil
  Modelling dependence with copulas and application to risk management
- E. Giacomini, W. Härdle, V. Spokoiny
  Inhomogeneous Dependency Modelling with Time Varying Copulae
  JPES, in print
- R. Nelsen
  An intoduction to copulas
  Springer, 1999
- O. Okhrin, Y. Okhrin, W. Schmid
  On the structure and estimation of hierarchical Archimedean copulas