Inhomogeneous Dependence Modelling with Time Varying Copulae

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Value-at-Risk

1. *value of linear portfolio* $w = (w_1, \ldots, w_d)^\top$ of assets $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$:

$$V_t = \sum_{j=1}^{d} w_j S_{j,t}$$

2. *profit and loss* (P&L) function:

$$\begin{aligned} L_{t+1} &= V_{t+1} - V_t = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1) \\ X_{t+1} &= \log S_{t+1} - \log S_t \end{aligned}$$

3. *Value-at-Risk* at level α:

$$VaR(\alpha) = F_L^{-1}(\alpha)$$

Log returns DCX & VW at 20030408

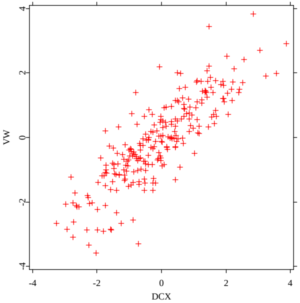


Figure 1: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20020415-20030408

Log returns DCX & VW at 20041027

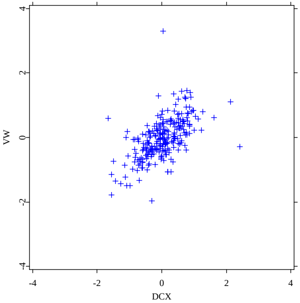


Figure 2: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20031103-20041027

VaR depends on the distribution F_X of log returns $X = (X_1, \ldots, X_d)^\top$.

1. How to model F_X and the dependence among $X_1, \ldots, X_d$?
2. How does F_X and the dependence among $X_1, \ldots, X_d$ vary over time ?

Traditional approach (*RiskMetrics*)

Log returns conditionally normal

$$X_t \sim N(0, \Sigma_t)$$

Drawbacks from multivariate normal distribution:

1. no heavy-tails
2. joint extreme values relatively infrequent
3. symmetry (elliptical distribution)

Copula based approach

Log returns distributed with copula C:

$$X \sim C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d); \theta\}$$

where $F_{X_1}, \dots, F_{X_d}$ are marginal distributions and θ is the copula (dependence) parameter.

Dynamic Copula Models

GARCH-residuals conditionally distributed with copula C:

$$\varepsilon_t \sim C\{F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d); \theta_t\}$$

1. change point analysis, structural breaks in dependence (Dias and Embrechts, 2004; 2006)
2. parametric specifications for dynamics of θ_t (Patton, 2006; Dias and Embrechts, 2007)
3. Markov switching regime (Rodriguez, 2007)

Adaptive Copulae

1. local parametric assumption: θ_t nearly constant on *homogeneity intervals*
2. for each t, *adaptively* find homogeneity interval
3. estimate dependence parameter *parametrically* from it

Estimate dependence parameter θ_t in a time varying interval

Local Parametric Assumption

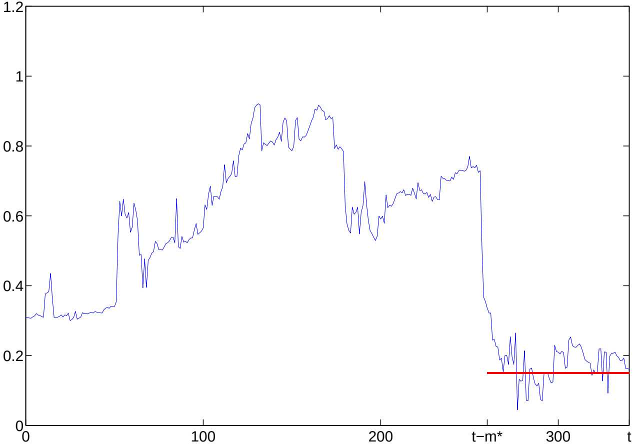
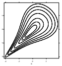


Figure 3: Parameter θ_t (blue), largest homogeneity interval at t (red).



Outline

1. Motivation ✓
2. Adaptive Copulae
3. Simulated Examples
4. Applications in Value-at-Risk
5. References
6. Appendix

Adaptive Copula Estimation I

1. nonparametric model

$$\mathcal{L}(\varepsilon_t|\mathcal{F}_{t-1}) = P_{\theta_t}$$

2. local parametric approximation: there exists a time interval I s.t.

$$\theta_t \approx \theta, \forall t \in I$$

Adaptive Copula Estimation II

1. small modelling bias condition (SMB)

$$\Delta_I(\theta) \quad = \quad \sum_{t \in I} \mathcal{K}(P_{\theta_t}, P_\theta) \leq \Delta$$

2. $I = [t_0 - m^*, t_0]$ is the *"oracle" choice*: largest interval where SMB holds
3. θ_{t_0} is ideally estimated from I by

$$\widetilde{\theta}_I = \operatorname*{argmax}_{\theta \in \Theta} L_I(\theta)$$

Adaptive Copula Estimation III

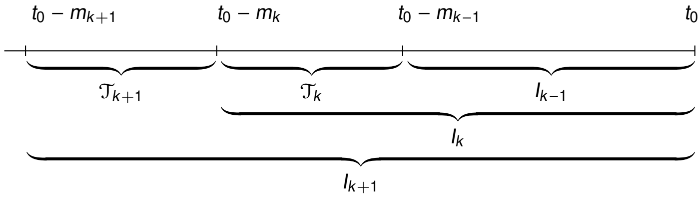
1. *"oracle" choice* depends on unknown parameter θ_t
2. adaptively estimate largest interval $\widehat{I}$ where homogeneity hypothesis is accepted
3. θ_{t_0} is estimated by

$$\widetilde{\theta}_{\widehat{I}} = \underset{\theta \in \Theta}{\operatorname{argmax}} \, L_{\widehat{I}}(\theta)$$

4. adaptive estimator: provide "oracle" quality, Spokoiny(2007)
5. *Local Change Point detection (LCP)*(Mercurio, Spokoiny, 2004): sequentially test $\theta_t = \theta$ within some interval

Local Change Point procedure (LCP)

1. define family of nested intervals $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ with length m_k as $I_k = [t_0 - m_k, t_0]$
2. define $\mathcal{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$



LCP

Start with $k = 1$ and

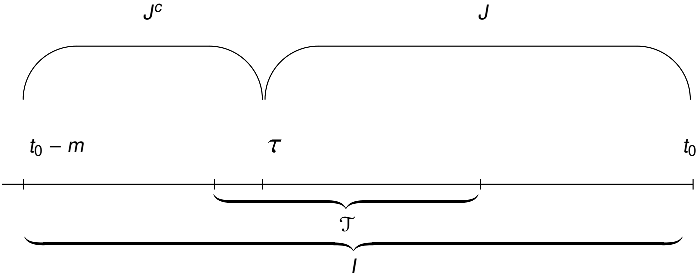
1. test homogeneity $H_{0,k}$ against change point alternative in $\mathcal{T}_k$ using I_{k+1};
2. if no change points in $\mathcal{T}_k$, accept I_k. Take $\mathcal{T}_{k+1}$ and repeat previous step until $H_{0,k}$ is rejected or largest possible interval I_K is accepted;
3. if $H_{0,k}$ is rejected in $\mathcal{T}_k$, homogeneity interval is the last accepted $\widehat{I} = I_{k-1}$
4. if largest possible interval I_K is accepted, $\widehat{I} = I_K$.

Test of homogeneity against a change point alternative

Interval $I = [t_0 - m, t_0]$, $\mathcal{T} \subset I$

$$H_0 \quad : \quad \forall \tau \in \mathcal{T}, \quad \theta_t = \theta \;\; \forall t \in J = [\tau, t_0], \quad \forall t \in J^c = I \setminus J$$

$$H_1 \quad : \quad \exists \tau \in \mathcal{T}, \quad \theta_t = \theta_1 \; \forall t \in J, \quad \theta_t = \theta_2 \neq \theta_1 \; \forall t \in J^c$$



Likelihood ratio test statistic for fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2} \{L_J(\theta_1) + L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= \widehat{L}_J + \widehat{L}_{J^c} - \widehat{L}_I \end{aligned}$$

Test statistic for unknown change point location:

$$T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}$$

Reject H_0 if for a critical value $\mathfrak{z}_I$

$$T_I > \mathfrak{z}_I$$

Adaptive Estimator $\widehat{\theta}(\mathfrak{z}_1, \ldots, \mathfrak{z}_K)$

Define the random sets

1. $$\mathcal{C}_k = \{T_k \leq \mathfrak{z}_k\}$$

2. I_k accepted:
$$\mathcal{A}_k = \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k$$

3. I_k accepted, I_{k+1} rejected:
$$\mathcal{B}_k = \mathcal{A}_k \backslash \mathcal{C}_{k+1}$$

Adaptive Estimator $\widehat{\theta}(\mathfrak{z}_1, \ldots, \mathfrak{z}_K)$

1. adaptive estimator $\widehat{\theta}_k$ at step k

$$\widehat{\theta}_k(\mathfrak{z}_1, \ldots, \mathfrak{z}_k) = \widetilde{\theta}_k \mathbf{1}(\mathcal{A}_k) + \sum_{l=0}^{k-1} \widetilde{\theta}_l \mathbf{1}(\mathcal{B}_l)$$

where $\widetilde{\theta}_k$ is weak (MLE) estimator at I_k

Choice of Critical Values $\mathfrak{z}$

In the parametric situation, $\theta_t = \theta^*$:

- desirable: $\widehat{\theta}_k = \widetilde{\theta}_k$
- "False alarm": the estimated homogeneity interval is too small and $\widehat{\theta}_k \neq \widetilde{\theta}_k$
- difference between $\widehat{\theta}_k$ and $\widetilde{\theta}_k$ measured by $L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_k) = L_{I_k}(\widetilde{\theta}_k) - L_{I_k}(\widehat{\theta}_k)$
- select $\mathfrak{z}_1, \ldots, \mathfrak{z}_K$ such that risk of the adaptive estimator $\widehat{\theta}_k$ of same order as of the non-adaptive estimator $\widetilde{\theta}_k$

Choice of Critical Values $\mathfrak{z}$

The critical values are $\mathfrak{z}_1, \ldots, \mathfrak{z}_K$ providing

$$\boldsymbol{E}_{\theta^*}\left|L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_k)\right|^{1/2} \leq \rho\Re(\theta^*), \qquad k = 1, \ldots, K, \qquad \theta^* \in \Theta \qquad (1)$$

where $0 < \rho \leq 1$ and $\Re(\theta^*)$ is the risk of the parametric estimation $\widetilde{\theta}_k$

$$\Re(\theta^*) = \max_{k \geq 1} \boldsymbol{E}_{\theta^*}\left|L_{I_k}(\widetilde{\theta}_k, \theta^*)\right|^{1/2}$$

Sequential Choice of $\mathfrak{z}$

Rewriting

$$\boldsymbol{E}_{\theta^*}\left|L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_k)\right|^{1/2} = \sum_{l=0}^{k-1} \boldsymbol{E}_{\theta^*}\left|L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_l)\right|^{1/2} \mathbf{1}(\mathcal{B}_l) \qquad (2)$$

Sequentially select $\mathfrak{z}_k$ for $k = 1, \ldots, K$ such that for $\theta^* \in \Theta$

$$\max_{k>l\geq 0} \boldsymbol{E}_{\theta^*}\left|L_{I_k}(\widetilde{\theta}_k, \widetilde{\theta}_l)\right|^{1/2} \mathbf{1}(\mathcal{B}_l) \leq \frac{\rho \Re(\theta^*)}{K} \qquad (3)$$

by Monte Carlo simulation under $H_0 : \theta_t = \theta^*, \forall t \in I_K$.

Simulated Examples
1. Clayton copula: sudden jump in dependence

Simulated sets of observations from 6-dimensional Clayton copula with parameter

$$\theta_t = \begin{cases} \vartheta_a & \text{if} \quad -390 \leq t \leq 10 \\ \vartheta_b & \text{if} \quad 10 < t \leq 210 \end{cases}$$

for $\vartheta_a, \vartheta_b = 0.1, 0.5, 0.75$ and 1.

Selection of I_k and $\mathfrak{J}_k$

1. set of numbers m_k defining the length of I_k and $\mathcal{J}_k$ in form of a geometric grid.
2. $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 = 20$ and $c = 1.25$ where $[x]$ means the integer part of x.
3. $I_k = [t_0 - m_k, t_0]$ and $\mathcal{J}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$.
4. $\rho = 0.2, 0.5, 1$
5. $\theta^* = 0.5, 1, 1.5$

Critical Values: weak dependence on θ^*

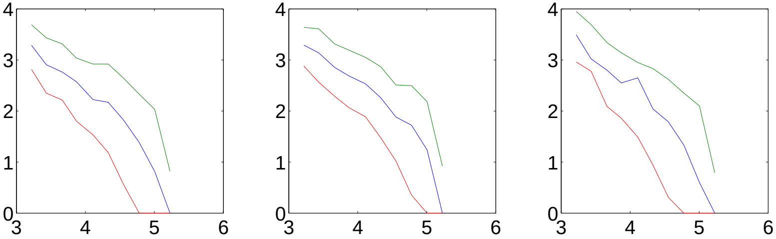


Figure 4: Critical values $\mathfrak{z}_k(\rho, \theta^*)$ (vertical axis), *log*(m_k) (horizontal axis) for $\rho = 0.2, 0.5$ and 1.0 (top to bottom), $\theta^* = 0.5, 1.0$ and 1.5 (left to right). Based on 5000 simulations from Clayton copula, $m_0 = 20$, $c = 1.25$, $k = 1, \ldots, 11$

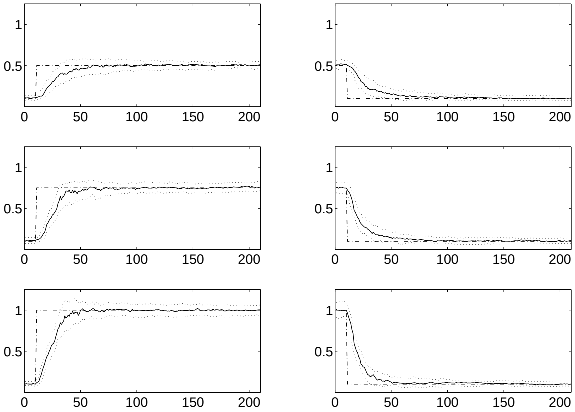


Figure 5: Pointwise median (full), 0.25 and 0.75 quantiles (dotted) from $\widehat{\theta_t}$. True parameter θ_t (dashed) with $\vartheta_a = 0.10$, $\vartheta_b = 0.50$, 0.75 and 1.00 (left, top to bottom) and $\vartheta_b = 0.10$, $\vartheta_a = 0.50$, 0.75 and 1.00 (right, top to bottom).

Based on 100 simulations from Clayton copula, estimated with LCP, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

Detection Delay

Detection delay δ at rule $r \in [0,1]$ to jump $\gamma = \theta_t - \theta_{t-1}$ at t

$$\delta(t,\gamma,r) = \min\{u \geq t : \widehat{\theta_u} = \theta_{t-1} + r\gamma\} - t$$

1. number of steps for estimated parameter to reach r-fraction of jump in real parameter
2. proportional to type II error (accept homogeneity when jump), decreasing in power and in $\mathcal{K}(H_0, H_1)$.

Detection Delay and Clayton Copula

For Clayton copula

1. KL is asymmetric: $\theta_0 < \theta_1, \mathcal{K}_d(\theta_0, \theta_1) > \mathcal{K}_d(\theta_1, \theta_0)$
2. detection delay δ is decreasing in $|\gamma|$ and higher for downward than for upward jumps.

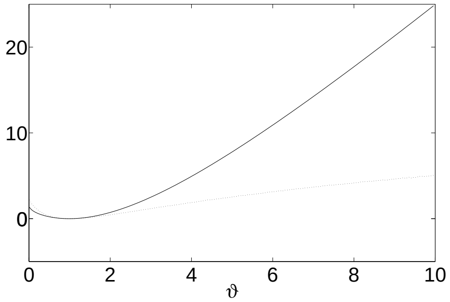


Figure 6: Kullback-Leibler divergences $\mathcal{K}(0.10, \vartheta)$ (full) and $\mathcal{K}(\vartheta, 0.10)$ (dashed), 6-dimensional Clayton copula

Detection Delay Statistics

$(\vartheta_a, \vartheta_b)$	r	mean	std dev.	max	min
$(0.50, 0.10)$	0.25	9.06	7.28	56	0
	0.50	13.64	9.80	60	0
	0.75	21.87	14.52	89	3
$(0.75, 0.10)$	0.25	5.16	4.24	21	0
	0.50	8.85	5.55	25	0
	0.75	16.72	10.37	64	3
$(1.00, 0.10)$	0.25	4.47	2.94	12	0
	0.50	7.94	4.28	22	0
	0.75	14.79	7.38	62	5
$(0.10, 0.50)$	0.25	8.94	6.65	36	0
	0.50	14.21	9.06	53	0
	0.75	21.43	12.15	68	0
$(0.10, 0.75))$	0.25	9.00	4.80	25	0
	0.50	14.30	5.96	40	3
	0.75	21.00	10.97	75	6
$(0.10, 1.00)$	0.25	7.39	3.67	19	0
	0.50	13.10	4.13	22	2
	0.75	20.13	7.34	55	10

Table 1: Statistics for detection delay δ at rule r, based on 100 simulations from Clayton copula, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

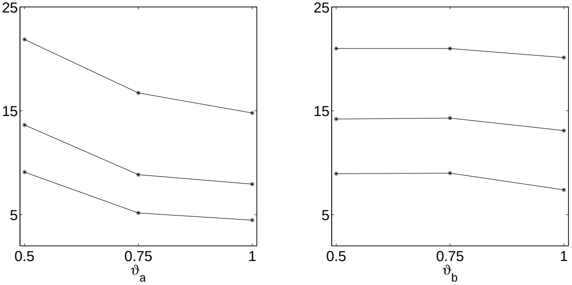


Figure 7: Mean detection delays (crosses) at $r = 0.75$, 0.50 and 0.25 from top to bottom. Left: $\vartheta_b = 0.10$(upward jump). Right: $\vartheta_a = 0.10$ (downward jump), Clayton Copula

2. Clayton copula: smooth change in dependence

Simulated sets of observations from 6-dimensional Clayton copula with parameter

$$\theta_t = \begin{cases} \vartheta_a & \text{if} \quad -350 \leq t \leq 50 \\ \vartheta_a + \frac{t-50}{100}(\vartheta_b - \vartheta_a) & \text{if} \quad 50 < t \leq 150 \\ \vartheta_b & \text{if} \quad 150 < t \leq 350 \end{cases}$$

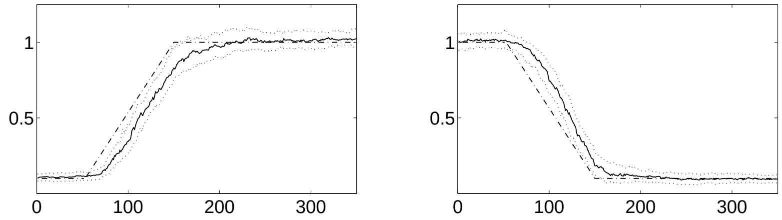


Figure 8: Pointwise median (full), 0.25, 0.75 quantiles (dotted) of estimated parameter $\widehat{\theta_t}$, true parameter θ_t (dashed). Based on 100 simulations, Clayton copula, $(\vartheta_a, \vartheta_b) = (1.00, 0.10)$, $m_0 = 20$ and $c = 1.25$

Application: Copulae and Value-at-Risk

The process $\{X_t\}_{t=1}^{T}$ of log-returns can be modelled (Chen and Fan, 2005) as

$$X_{j,t} = \sigma_{j,t}\varepsilon_{j,t}$$

with

$$\sigma_{j,t}^2 = E[X_{j,t}^2 \mid \mathcal{F}_{t-1}]$$

where $\mathcal{F}_t$ is the available information at time t.

The standardized innovations

$$\varepsilon_t = (\varepsilon_{1,t}, \dots, \varepsilon_{d,t})^\top$$

are independent with distribution function

$$F_{\varepsilon_t}(x_1, \dots, x_d) = C\{F_{t,1}(x_1), \dots, F_{t,d}(x_d); \theta_t\}$$

where

1. C is copula with parameter θ
2. ε_j have continuous marginal distributions F_j, $j = 1, \dots, d$

Copulae and Value-at-Risk II

For log-returns $\{x_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$, estimation of VaR at level α:

1. determine innovations $\widehat{\varepsilon}_t$
2. specificy and estimate marginal distributions $F_j(\widehat{\varepsilon}_j)$
3. specificy a copula C and estimate dependence parameter θ
4. simulate innovations ε and losses L
5. determine $\widehat{VaR}(\alpha)$, the empirical α-quantile of F_L.

Empirical Results

Estimation of VaR from portfolios composed of 6 DAX stocks. Use 2 groups:

- group 1: Volkswagen, DaimlerChrysler, Allianz, Münchener Rückversicherung, Bayer and BASF (*high concentration*)
- group 2: Siemens, E.ON, ThyssenKrupp, Lufthansa, Schering and Henkel (*low concentration*)
- closing daily prices from 01.01.2000 to 31.12.2004 (1270 observations)
- data available in `http://sfb649.wiwi.hu-berlin.de/fedc`

VaR Estimation

1. log-returns from stock j modelled by

$$X_{t,j} = \sigma_{t,j}\varepsilon_{t,j}$$

2. $\sigma^2_{t,j}$ estimated at time t by exponential smoothing with $\lambda = \frac{1}{20}$

$$\widehat{\sigma}^2_{t,j} = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X^2_{s,j}$$

3. empirical distribution of the obtained residuals $\widehat{\varepsilon}_{t,j}$
4. 6-dim copula belongs to Clayton family

Backtesting

The *exceedances ratio* and *relative exceedance error* at level α for a portfolio w are given by

$$\widehat{\alpha}_w(\alpha) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}\{l_t < \widehat{VaR}_t(\alpha)\}$$

$$e_w(\alpha) = \frac{\widehat{\alpha}_w - \alpha}{\alpha}$$

where $\{l_t\}$ are realizations of the P&L function.

Exceedance Ratio - Exceedance Error

Calculated for set of portfolios

$$\mathcal{W} = \{w^*, w_n; n = 1, \dots, 100\}$$

where

1. $w^* = (w_1^*, \dots, w_6^*)^\top$, is the equally weighted portfolio $w_i^* = \frac{1}{6}$, $i = 1, \dots, 6$,
2. w_n is a realization of a random vector uniformly distributed on $\mathcal{S} = \{(x_1, \dots, x_6) \in \mathbb{R}^6 : \sum_{i=1}^6 x_i = 1, x_i \geq 0.1\}$

with *RiskMetrics* (RM), Clayton copula with moving window (MW) and Local Change Point (LCP) estimation.

VaR Estimation: Performance

The performances in VaR estimation are compared based on

1. average exceedance error

$$A_{\mathcal{W}} = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} e_w$$

2. standard deviation

$$D_{\mathcal{W}} = \left\{ \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} (e_w - A_{\mathcal{W}})^2 \right\}^{1/2}$$

Results Group 1

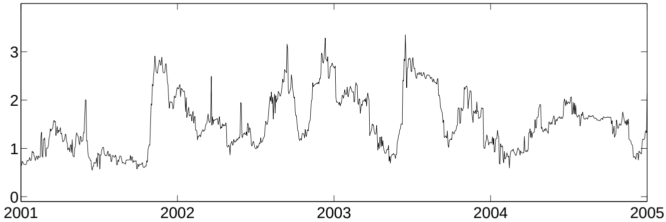


Figure 9: Estimated copula parameter $\widehat{\theta}_t$ for group 1, LCP method, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$, Clayton copula

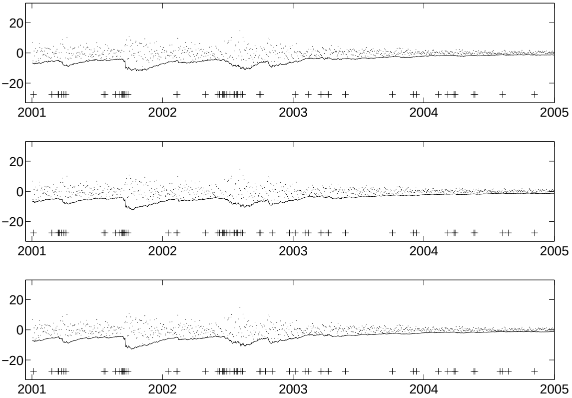


Figure 10: P&L (dots), Value-at-Risk at level $\alpha = 0.05$ (line), exceedances (crosses), estimated with LCP (above), MW (middle) and RM (below), for equally weighted portfolio w^*, group 1

	RM		MW		LCP		
	α						
	5.00	1.00	5.00	1.00	5.00	1.00	$(\times 10^{-2})$
$\widehat{\alpha}_{w^*}$	6.11	1.48	5.62	0.59	5.52	0.69	
$\widehat{\alpha}_{w_1}$	5.91	1.38	5.42	0.49	5.42	0.69	$(\times 10^{-2})$
$\widehat{\alpha}_{w_2}$	6.40	1.28	5.91	0.49	5.71	0.59	
$A_{\mathcal{W}}$	0.23	0.45	0.11	−0.49	0.11	−0.36	
$D_{\mathcal{W}}$	0.04	0.14	0.06	0.08	0.06	0.10	

Table 2: Exceedance ratios for portfolios w^*, w_1 and w_2, average and standard deviation from relative exceedance errors across levels and methods, group 1

Results Group 2

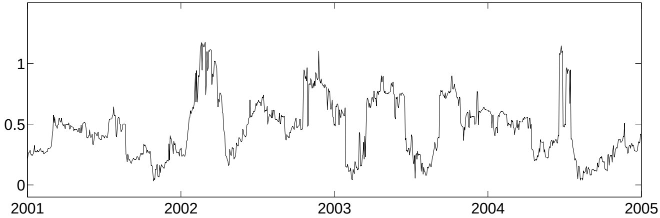


Figure 11: Estimated copula parameter $\widehat{\theta}_t$ for group 2, LCP method, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$, Clayton copula

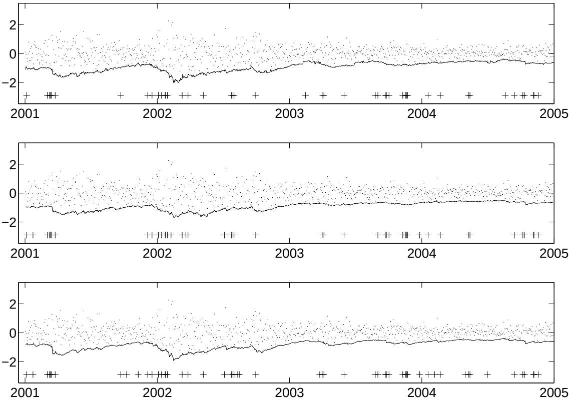


Figure 12: P&L (dots), Value-at-Risk at level $\alpha = 0.05$ (line), exceedances (crosses), estimated with LCP (above), MW (middle) and RM (below) for equally weighted portfolio w^*, group 2

	RM		MW		LCP		
	α						
	5.00	1.00	5.00	1.00	5.00	1.00	$(\times 10^{-2})$
$\widehat{\alpha}_{w^*}$	5.42	1.58	4.53	0.39	4.53	0.30	
$\widehat{\alpha}_{w_1}$	5.81	1.77	5.02	0.39	5.02	0.39	$(\times 10^{-2})$
$\widehat{\alpha}_{w_2}$	5.62	1.58	5.12	0.39	5.22	0.30	
$A_{\mathcal{W}}$	0.16	0.57	−0.10	−0.65	−0.09	−0.65	
$D_{\mathcal{W}}$	0.04	0.16	0.06	0.09	0.06	0.08	

Table 3: Exceedance ratios for portfolios w^*, w_1 and w_2, average and standard deviation from relative exceedance errors across levels and methods, group 2

Outlook

Adaptive copulae estimation based on local constant assumption.
Further research:

- time varying dependence: local parametric copulae
- different copulae families
- adaptive tail dependence

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Appendix

1. *Small modelling bias (SMB)* condition for an interval I,

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta,$$

2. For $\Delta > 0$ and set of intervals $\{I_k\}_{k=1,\ldots,K}$:

$$k^* = \{s \in \{1, \ldots, K\} : k \leq s, \Delta_{I_k}(\theta) \leq \Delta\}$$

Theorem 1 (parametric case):

Under the local parametric assumption θ can be estimated by $\widetilde{\theta}_{I_k}$ in an interval I_k such that it holds:

$$E_{\mathbb{P}_\theta} \left| L_{I_k} \left(\widetilde{\theta}_k, \theta \right) \right|^r \leq \mathcal{R}_{2r}$$

- estimation risk is bounded by constant $\mathcal{R}_{2r}$
- in the Gaussian regression case $\mathcal{R}_{2r} = E|\xi|^{2r}$ where $\xi \sim N(0,1)$

Theorem 2 (nonparametric case):

Under the *SMB* condition (i.e. $\Delta_{I_k}(\theta) \leq \Delta$) it holds

$$E_{\mathbb{P}} \left| L_{I_k} \left(\widetilde{\theta}_k, \theta \right) \right|^{r/2} \leq \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta),$$

i.e. $\widetilde{\theta}_{I_k}$ is a "good" estimator of θ in an interval I_k.

- $\exp(\Delta)$ is payment for approximation

Sequential choice of critical values

Choose λ_{I_k} such that for $k = 1, \ldots, K$

$$E_{\mathbb{P}_\theta} \left| L_{I_k} \left(\widetilde{\theta}_k , \widehat{\theta}_k \right) \right|^r \leq \rho \mathcal{R}_{2r}$$

i.e. in the parametric case the risk of adaptive estimate is of same order as risk of non-adaptive "oracle" estimate $\widetilde{\theta}_{I_K}$:

$$E_{\mathbb{P}_\theta} \left| L_{I_K} \left(\widetilde{\theta}_K , \widehat{\theta} \right) \right|^r \leq \rho \mathcal{R}_{2r}$$

Theorem 3 (critical values):

For λ_{I_k}, $k = 1, \ldots, K$ such that

$$E_{\mathbb{P}_\theta} \left| L_{I_k} \left(\widetilde{\theta}_k, \widehat{\theta}_k \right) \right|^r \leq \rho \mathcal{R}_{2r} \tag{4}$$

there exist ι_0, ι such that

$$\lambda_{I_k} \leq \iota_0 \log K + \iota (K - k) \tag{5}$$

- simplified procedure: select ι_0, ι such that (4) holds

Propagation to nonparametric:

Under the *SMB* condition, i.e. $\Delta_{I_k}(\theta) \leq \Delta$ for $k \leq k^*$, it follows from theorem 2 for $\rho > 0$

$$
\begin{aligned}
E_{\mathbb{P}} \left| L_{I_{k^*}} \left(\widetilde{\theta}_{k^*}, \theta \right) \right|^{r/2} & \leq \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta) \\
E_{\mathbb{P}} \left| L_{I_{k^*}} \left(\widetilde{\theta}_{k^*}, \widehat{\theta}_{k^*} \right) \right|^{r/2} & \leq (\rho \mathcal{R}_{2r})^{1/2} \cdot \exp(\Delta)
\end{aligned}
$$

- as long as SMB holds, local parametric approach is justified and adaptive estimate behaves similarly to ”oracle” estimate

Theorem 4 (stability after propagation):

Adaptive estimator $\widehat{\theta}$ provides same quality of estimation as non-adaptive oracle estimator $\widetilde{\theta}_{I_{k^*}}$:

$$E_{\mathbb{P}} \left| L_{I_{k^*}} \left(\widetilde{\theta}_{k^*}, \widehat{\theta} \right) \right|^{r/2} \leq const \cdot \left\{ \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta) + \mathfrak{z}_{k^*}^{r/2} \right\}$$

- $\exp(\Delta)$ is payment for approximation
- $\mathfrak{z}_{k^*}$ is payment for adaptation