Measuring Conditional Dependence

Dependence of Stock Returns in Bull and Bear Markets

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Notation

X, Y random variables with joint distribution function

$$F_{X,Y} = F$$

and continuous marginal distribution functions

$$F_X = G \qquad \text{and} \qquad F_Y = H.$$

$X_1, ..., X_d$ random variables with joint distribution function

$$F_{X_1,...,X_d}$$

and continuous marginal distribution functions

$$F_{X_i}, i = 1, ..., d.$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

For $d = 2$ we have

$$\rho = \frac{cov(F_X(X), F_Y(Y))}{\sqrt{var(F_X(X))}\sqrt{varF_Y(Y))}} = \frac{cov(U, V)}{\sqrt{var(U)}\sqrt{var(V)}}$$

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$$= \frac{\int_0^1\int_0^1 uvdC(u, v) - \left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} = 12\int_0^1\int_0^1 uvdC(u, v) - 3$$

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$$= \frac{\int_0^1\int_0^1 C(u,v)dudv - \frac{1}{4}}{\frac{1}{3}-\frac{1}{4}} = \frac{\int_0^1\int_0^1 C(u,v)dudv - \int_0^1\int_0^1 uvdudv}{\int_0^1\int_0^1 \min\{u,v\}\,dudv - \int_0^1\int_0^1 uvdudv}$$

1 Conditional Versions of Spearman's Rho

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For $d \geq 2$ let

$$\rho = \frac{\int_{[0,1]^d} C(u)du - \int_{[0,1]^d} u_1 \cdot \ldots \cdot u_d du}{\int_{[0,1]^d} \min\{u\}\, du - \int_{[0,1]^d} u_1 \cdot \ldots \cdot u_d du}$$

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$$= \frac{\int\limits_{[0,1]^d} C(u)du - \left(\frac{1}{2}\right)^d}{\frac{1}{d+1} - \left(\frac{1}{2}\right)^d}$$

$$= \frac{d+1}{2^d - (d+1)}\left(2^d \int\limits_{[0,1]^d} C(u)du - 1\right)$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

$$\rho(p) = \frac{\int\limits_{[0,p]^d} C(u)du - \int\limits_{[0,p]^d} u_1 \cdot \ldots \cdot u_d du}{\int\limits_{[0,p]^d} \min\{u\} du - \int\limits_{[0,p]^d} u_1 \cdot \ldots \cdot u_d du}$$

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$$= \frac{\int_{[0,p]^d} C(u)du - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d}$$

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$$= \frac{\int_{[0,p]^d} C(u)du - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d}$$

$$\rho_L = \lim_{p\downarrow 0} \rho(p) = \lim_{p\downarrow 0} (d+1)\frac{1}{p^d+1} \int_{[0,p]^d} C(u)du$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

Relation between ρ_L and

$$\lambda_L = \lim_{p \downarrow 0} \frac{C(p, p)}{p}$$

for $d = 2$.

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Proposition

$$\lambda_L \leq \rho_L \leq \min\{1, 2\lambda_L\}$$

and

$$\begin{aligned} \lambda_L &= 0 \Leftrightarrow \rho_L = 0 \\ \lambda_L &= 1 \Leftrightarrow \rho_L = 1. \end{aligned}$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

Let $X_1, X_2, ..., X_n$ i.i.d. from $X = (X_1, ..., X_d)$

$$\widehat{\rho}_n(p) = \frac{\int\limits_{[0,p]^d} \widehat{C}_n(u)du - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d}$$

where

$$\widehat{C}_n(u) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} 1_{\{\widehat{U}_{ij,n} \leq u_i\}}$$

and

$$\widehat{U}_{ij,n} = \widehat{F}_{X_{i,n}}(X_{ij}) = \frac{1}{n}(\text{rank of } X_{ij} \text{in } X_{i1}, ..., X_{in}).$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

Asymptotic Normality of $\widehat{\rho}_n(p)$

$$\sqrt{n}(\widehat{\rho}_n(p) - \rho(p)) \Rightarrow \mathbb{G}(p)$$

where

$$\mathbb{G}(p) = \int_{[0,p]^d} \mathbb{G}_C(u)du \Big/ \left(\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2} \right)^2 \right)$$

and

$$\begin{aligned} \mathbb{G}_C(u) &= \mathbb{B}_C(u) - \sum_{i=1}^{d} D_i C(u) \mathbb{B}_C(u^{(i)}) \\ u^{(i)} &= (1, ..., 1, u_i, 1, ..., 1). \end{aligned}$$

1 Conditional Versions of Spearman's Rho

1.1 Based on Original Copula

$\mathbb{B}_C$ is a centered Gaussian Process with

$$E\left\{\mathbb{B}_C(u)\mathbb{B}_C(v)\right\} = C\left(u \wedge v\right) - C(u)C(v)$$

The covariance structure of $\mathbb{G}(p)$ is given by

$$E(\mathbb{G}(p)\mathbb{G}(q)) = \frac{1}{c(p) \cdot c(q)} \int\limits_{[0,p]^d \times [0,q]^d} E\left\{\mathbb{G}_C(u) \cdot \mathbb{G}_C(v)\right\} duv,$$

$0 < p, q \leq 1$, with

$$c(p) = \frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d.$$

The Bootstrap

$\left(X_j^B\right)_{j=1,\ldots,n}$ denotes the bootstrap sample which is obtained by sampling from $(X_j)_{j=1,\ldots,n}$ with replacement. $\widehat{\rho}_n^B(p)$ denotes the corresponding bootstrap estimator. Then the process

$$\sqrt{n}\left(\widehat{\rho}_n^B(p) - \widehat{\rho}_n(p)\right)$$

converges weakly to the same Gaussian process as

$$\sqrt{n}(\widehat{\rho}_n(p) - \rho(p))$$

with probability one (under the above stated assumptions).

For $0 < p < 1$ let

$$A_{L,p} = \{(x, y) : x \leq G^{-1}(p), y \leq H^{-1}(p)\}$$

and

$$P((X, Y) \in A_{L,p}) = C(p, p) > 0$$

consider

$$\begin{aligned} F_{L,p}(x, y) &= P(X \leq x, Y \leq y \mid (X, Y) \in A_{L,p}) \\ &= \frac{C(G(x \wedge G^{-1}(p))), H(y \wedge H^{-1}(p)))}{C(p, p)} \end{aligned}$$

with continous marginals

$$G_{L,p}(x) = \frac{C(G(x \wedge G^{-1}(p)), p)}{C(p, p)}$$

and $H_{L,p}(x)$ similiarly.

There exists a unique copula $C_{L,p}$ with

$$C_{L,p}(u, v) = F_{L,p}\left(G_{L,p}^{-1}(u), H_{L,p}^{-1}(v)\right).$$

Conditional (lower tail) copula at level p.

Conditional (lower tail) Spearman's rho at level p is defined by

$$\rho_{L,p} = 12 \int_{[0,1]^2} uvC_{L,p}(u, v) - 3.$$

Conditional (lower tail) Spearman's rho at level p is defined by

$$\rho_{L,p} = 12 \int_{[0,1]^2} uvC_{L,p}(u,v) - 3.$$

Analogously: Conditional (upper tail) copula at level p, $C_{U,p}$ and conditional (upper tail) Spearman's rho at level p

$$\rho_{U,p} = 12 \int_{[0,1]^2} uvC_{U,p}(u,v) - 3.$$

2 Dependence of Stock Returns in Bull and Bear Markets

2.1 A Testing Procedure

Testing problems

$$H_0 : \rho_{L,p} = \rho_{U,p}$$

$$H_1 : \rho_{L,p} \neq \rho_{U,p}$$

or

$$H_0 : \rho_{L,p} \leq \rho_{U,p}$$

$$H_1 : \rho_{L,p} > \rho_{U,p}.$$

2 Dependence of Stock Returns in Bull and Bear Markets

2.1 A Testing Procedure

$(X_1, Y_1), \ldots, (X_n, Y_n)$ i.i.d. from (X, Y)

$$\widehat{A}_{L,p} := \left\{(x, y) : x \leq \widehat{G}_n^{-1}(p), y \leq \widehat{H}_n^{-1}(p)\right\}$$

and $n_L := \left|\widehat{A}_{L,p}\right|$

$$\widehat{\rho}_{L,n} = \frac{12}{n_L} \sum_{i \in I_{\widehat{A}_{L,p}}} \frac{r_{L,n}(X_i)}{n_L} \frac{r_{L,n}(Y_i)}{n_L} - 3$$

$I_{\widehat{A}_{L,p}}$ set of indices i where $(X_i, Y_i) \in \widehat{A_{L,p}}$.

$r_{L,n}(\cdot)$ rank of one marginal observation relative to all other observations in $\widehat{A_{L,p}}$.

$\widehat{\rho}_{U,n}$ is similarly defined.

Testing procedure

1. Compute $\widehat{\rho}_{L,n}$and $\widehat{\rho}_{U,n}$ from the observations in $\widehat{A}_{L,p}$ and $\widehat{A}_{U,p}$, where p is fixed with $p \leq \frac{1}{2}$.

Testing procedure

1. Compute $\widehat{\rho}_{L,n}$and $\widehat{\rho}_{U,n}$ from the observations in $\widehat{A}_{L,p}$ and $\widehat{A}_{U,p}$, where p is fixed with $p \leq \frac{1}{2}$.
2. Compute N_B bootstrap replications of $\widehat{\rho}_{L,n}$and $\widehat{\rho}_{U,n}$ from the observations in $\widehat{A}_{L,p}$ and $\widehat{A}_{U,p}$ and calculate the corresponding estimates for the asymptotic variances of σ_L^2 and σ_U^2, say $\widehat{\sigma}_L^2$ and $\widehat{\sigma}_U^2$.

Testing procedure (continued)

3. Reject $H_0 : \rho_{L,p} = \rho_{U,p}$ if

$$\left| \frac{\widehat{\rho}_{L,n} - \widehat{\rho}_{U,n}}{\sqrt{\widehat{\sigma}_L^2 / n_L + \widehat{\sigma}_U^2 / n_U}} \right| \geq \Phi^{-1}\left(1 - \frac{\alpha}{2}\right),$$

where $\alpha > 0$ is a small error probability of the first kind and Φ denotes the distribution function of the standard normal distribution.

Data: Daily Returns 1992 − 03 − 02 to 2002 − 03 − 01
Number of observation $n = 2523$
21 stocks of the German DAX 30 and DAX 30 index.

	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
$\overline{\widehat{\rho}}_L$	.3609	.3009	.3149	.3387	.3479
$\overline{\widehat{\rho}}_U$	.2685	.2409	.2400	.2571	.2749
$\overline{\widehat{\rho}_L - \widehat{\rho}_U}$	.0924	.0600	.0749	.0816	.0730
$\overline{\lvert\widehat{\rho}_L - \widehat{\rho}_U\rvert}$	.1528	.0891	.0829	.0840	.0754

Average conditional rank correlation coefficients, differences, and absolute differences of all 231 asset combinations for different threshold probabilities $p = q$.

Number of rejections for the different hypothesis tests, various threshold probabilities $p = q$, and error probabilities α given 231 asset combinations. Further, the numbers of asset combinations where $\widehat{\rho}_L$ is larger or smaller than $\widehat{\rho}_U$ (last rows of Panel 2 and Panel 3).

Panel 1	$H_0 : \rho_{L,p} = \rho_{U,p}$ vs. $H_1 : \rho_{L,p} \neq \rho_{U,p}$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	37	34	67	89	107
0.05	25	16	51	67	72
0.01	6	3	19	31	30

Panel 2	$H_0 : \rho_{L,p} \leq \rho_{U,p}$ vs. $H_1 : \rho_{L,p} > \rho_{U,p}$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	55	61	94	136	148
0.05	35	32	67	89	107
0.01	11	4	30	42	50
$\widehat{\rho}_L > \widehat{\rho}_U$	170	173	196	218	217

Panel 2	$H_0: \rho_{L,p} \leq \rho_{U,p}$ vs. $H_1: \rho_{L,p} > \rho_{U,p}$				
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Panel 3	$H_0: \rho_{L,p} \geq \rho_{U,p}$ vs. $H_1: \rho_{L,p} < \rho_{U,p}$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	9	7	3	0	0
0.05	2	2	0	0	0
0.01	1	0	0	0	0
$\hat{\rho}_L < \hat{\rho}_U$	61	58	35	13	14

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